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⑥ SOME BAYESIAN INFERENCE PROCEDURES
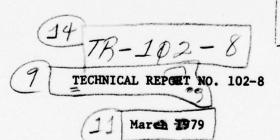FOR USE IN DEVELOPING AN IMPACT
ACCELERATION INJURY PREDICTION MODEL

by

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and
Dennis E./Smith

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I. INTRODUCTION

The U.S. Navy's impact acceleration research program being conducted by the Naval Aerospace Medical Research Laboratory (NAMRL) Detachment is engaged in experimentation concerning dynamic response of the human and simian head/neck system as a function of anthropometric and acceleration parameters. NAMRL is accumulating data on dynamic response of the head/neck body segment for both humans and simians. Research on the development of a model of injury probability based primarily on information in the data is contained in three Desmatics technical reports [3, 4, 5]. The development of a model of injury that utilizes an experimenter's prior knowledge about injury probability in addition to the sample information may allow for more accurate inferences to be made concerning the likelihood of injury as a function of dynamic and physical variables.

This technical report describes procedures for incorporating prior knowledge with the statistical information. Because this report is an extension of the inference procedures found in a previous technical report [2], it is suggested that report be read before proceeding further with this one. The logistic models and simulated data used in the present report were discussed previously [4]. Since the data was simulated, the true underlying model parameters are known. This allows an assessment of the accuracy of the inference procedures derived. As with inferences previously discussed [2], the inference of foremost importance in this study is the prediction of threshold levels which allow for only a small, prespecified chance of head/neck injury.

II. BACKGROUND

Many times a researcher may want to quantitatively incorporate his or her own prior knowledge about an experiment with data to be generated by the experiment. This can effectively be accomplished by the Bayesian approach to statistical inference. Through this approach, the experimenter also determines the importance his prior information should have with respect to the future sample information. The Bayesian approach uses subjective information before the experimental data is gathered to help prevent any subjective bias that could result due to observation of the data. For further discussion on this concept, see [6].

Bayesian inference quantitatively incorporates prior information with sample information by the use of a prior probability distribution. This distribution can be used to quantify the experimenter's prior beliefs and expert knowledge about the experiment. Once a prior distribution is chosen, the inference procedure follows from Bayes' Theorem in a straightforward manner. The problem of assessing the prior distribution for probability of injury is treated in the following paragraphs, which discuss how to effectively combine an experimenter's prior knowledge with the sample information in the form of maximum likelihood estimates obtained previously [2].

The assessment of a prior distribution for the parameter(s) of interest must be done in such a way as to quantify the experimenter's knowledge and intuition as accurately as possible. The prior distribution should not be too complex and should be experimentally interpretable. For

the multiparameter logistic model, it is unlikely that the experimenter will be able to accurately choose a multivariate prior distribution for $\underline{\beta}$, the vector of coefficients. However for a given critical $\underline{x}_0$ vector of experimental conditions, the experimenter should have an intuitive idea of the value of the probability[1] of injury, $p(\underline{x}_0)$. The Bayesian inference procedures in this technical report employ a prior distribution for $p(\underline{x}_0)$ with $\underline{x}_0$ given. If $p(\underline{x}_0)$ is to be considered the outcome of a random variable, $\tilde{p}(\underline{x}_0)$, then it is reasonable to assume that the outcomes of $\tilde{p}(\underline{x}_0)$ should be distributed smoothly around some prior estimate of $p(\underline{x}_0)$. This suggests the use of the beta family of prior distributions for $\tilde{p}(\underline{x}_0)$. The beta probability density function is

$$g(p) = \frac{\Gamma(n')}{\Gamma(r')\Gamma(n' - r')} \, p^{r' - 1}(1 - p)^{n' - r' - 1}, \ 0 < p < 1,$$

where Γ is the gamma function and r' and n' are experimentally interpretable parameters. For future clarity all parameters of prior distributions will be primed.

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[1]
    For greater clarity in algebraic expressions, this probability is denoted by a lower case p rather than an upper case P, as in previous reports [2, 3, 4, 5].

III. ASSESSING A PRIOR DISTRIBUTION

There are several ways to choose the proper parameters for the beta prior. For example, see reference [6]. This paragraph describes a method that is both intuitive and likely to lead the experimenter into giving the proper weight to his or her prior information. If $\tilde{p}(\underline{x}_0)$ has a beta distribution with parameters r' and n', then

$$E[\tilde{p}(\underline{x}_0)] = r'/n'.$$

In a binomial experiment, if n independent, identical trials are conducted and r "successes" are observed, then the estimate of p is r/n.

The expression,

$$E[\tilde{p}(\underline{x}_0)] = r'/n',$$

has the interpretation that the experimenter feels his or her prior information is worth n' experimental trials and that in n' trials at $\underline{x}_0$, r' injuries would be expected. In other words, the experimenter feels that his or her information is worth n' sample data observations and expects the true probability of injury to be r'/n' at $\underline{x}_0$. For example, if the experimenter feels, in a particular case, that prior information is worth 10 sample observations and the prior estimate of $p(\underline{x}_0)$ is .1, then a beta prior with parameters r' = 1 and n' = 10 should be used.

Suppose the experiment is conducted at $\underline{x}_0$, N independent times. Let y be the number of injuries in the N trials. Then y has the binomial

probability distribution. Using a beta prior for $\tilde{p}(\underline{x}_0)$ and the binomial distribution for y it is easy to compute $E[\tilde{p}(\underline{x}_0)|y]$, called the posterior expectation of $\tilde{p}(\underline{x}_0)$, which is an estimate of $p(\underline{x}_0)$ that employs both sample information and prior information. $E[\tilde{p}(\underline{x}_0)|y]$ is computed from the posterior probability density function, which in turn is computed from the binomial distribution for y and the beta prior distribution. The fact that y has a binomial distribution and the prior density is beta implies that the posterior probability density for $\tilde{p}(\underline{x}_0)$ is beta with parameters $r'' = y + r'$ and $n'' = N + n'$. (See [6].) Throughout this report posterior distribution parameters are double primed. Thus,

$$E[\tilde{p}(\underline{x}_0)|y] = r''/n''.$$

Of course, the experiment is never actually repeated at $\underline{x}_0$. The sample information is scattered over the region of the $\underline{x}$-space. Let the sample size used to form the binary regression estimate, $\hat{p}(\underline{x})$, be N. Had the experiment been repeated N times at $\underline{x}_0$, more information on $p(\underline{x}_0)$ would be available than is contained in

$$\hat{p}(\underline{x}_0) = [1 + \exp(-\underline{x}_0'\hat{\underline{\beta}})]^{-1},$$

where $\hat{\underline{\beta}}$ is the maximum likelihood estimate of $\underline{\beta}$. (For further discussion on estimation of $p(\underline{x}_0)$ by the maximum likelihood method see [2].) Thus, the variability of y/N should be less than that of $\hat{p}(\underline{x}_0)$ if the experiment were in actuality repeated N times at $\underline{x}_0$.

IV. BAYESIAN INFERENCES

The posterior distribution could be approximated by substituting $N\hat{p}(\underline{x}_0)$ for y in the expression for r". However, then the amount of sample information would be over-represented because the sample information in $\hat{p}(\underline{x}_0)$ is less than that in y obtained by repeating the experiment N times at $\underline{x}_0$. To adjust for this over-representation of sample information at $\underline{x}_0$, a number, N^*, can be found such that the sample information obtained by replicating the experiment N^* times at $\underline{x}_0$ will equal the sample information represented by $\hat{p}(\underline{x}_0)$. N^* in this technical report is referred to as the effective sample size.

To compute such a number, N^*, find an N^* such that

$$\text{Var}(y/N^*) = \text{Var}[\hat{p}(\underline{x}_0)].$$

Both variances cannot be found exactly, but reasonable estimates can be found for each. By expanding

$$\hat{p}(\underline{x}) = [1 + \exp(-\underline{x}_0'\hat{\underline{\beta}})]^{-1}$$

in a first order Taylor series about $\underline{x}_0'\underline{\beta}$ and then substituting $\hat{\underline{\beta}}$ for $\underline{\beta}$, an approximate estimate for $\text{Var}[\hat{p}(\underline{x})]$ can be found. Thus, since

$$\text{Var}[\hat{p}(\underline{x}_0)] \cong (\hat{p}(\underline{x}_0)[1 - \hat{p}(\underline{x}_0)])^2\underline{x}_0'\hat{\underline{\Sigma}}\underline{x}_0$$

where $\hat{\underline{\Sigma}}$ denotes the estimated covariance matrix of $\hat{\underline{\beta}}$, and

$$\text{Var}(y/N^*) \cong \hat{p}(\underline{x}_0)[1 - \hat{p}(\underline{x}_0)]/N^*,$$

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this implies that

$$N^* \cong (\hat{p}(\underline{x}_0)[1 - \hat{p}(\underline{x}_0)]\underline{x}_0'\hat{\underline{\Sigma}}\underline{x}_0)^{-1}.$$

N^* represents the amount of sample information that can be utilized in a binary regression set up by specifying a prior distribution for the probability of injury only at $\underline{x}_0$.

Since the effective sample size is N^* and since y is approximated by $N^*\hat{p}(\underline{x}_0)$, the posterior distribution (which is beta with parameters $r'' = y + r'$ and $n'' = N^* + n'$) can be approximated by a beta distribution with parameters $r'' = N^*\hat{p}(\underline{x}_0) + r'$ and $n'' = N^* + n'$. Thus

$$E[\tilde{p}(\underline{x}_0)|y] \cong (N^*\hat{p}(\underline{x}_0) + r')/(N^* + n')$$

is the Bayes point estimate of $p(\underline{x}_0)$. Through Bayesian theory, it can also be shown that $E[\tilde{p}(\underline{x}_0)|y]$ is the probability of a future injury at $\underline{x}_0$ given the present effective sample size and prior information.

A. INTERVAL ESTIMATION

All Bayesian inferences are made by using the posterior distribution. To compute a Bayesian $100(1 - \alpha)$% credible interval[1] for $p(\underline{x}_0)$, c_1 and c_2 must be found such that $Pr[c_1 \leq \tilde{p}(\underline{x}_0) \leq c_2|y] = 1 - \alpha$. As previously shown, the posterior distribution can be approximated by a beta distribution with parameters $r'' = N^*p(\underline{x}_0) + r'$ and $n'' = N^* + n'$. Thus, by using a beta distribution with parameters r'' and n'', approximate values for c_1 and c_2 may be calculated. Computation of c_1 and c_2 involves evaluation of the inverse incomplete beta function. This function is

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[1] The analog of a classical confidence interval.

tabled in [6], for example. The values of c_1 and c_2 can also be computed from percentiles of the F-distribution, as described in [1]. Figure 1 gives examples of Bayesian credible intervals and their corresponding maximum likelihood estimate confidence intervals for models A and B using sample sizes 100 and 1000 for each model. See [4, 5] for a description and discussion of these models. The critical $\underline{x}_0$ vector used in these examples was:

$$\underline{x}_0 = (-0.0862, -0.2621, 0.9114, -0.7660, -0.1363, -0.4655).$$

B. HYPOTHESIS TESTING

There is no standard method of Bayesian hypothesis testing. This technical report will use a Bayesian hypothesis testing technique that employs the ratio of posterior probability of the null hypothesis being true to the posterior probability of the alternative hypothesis being true. The hypotheses will be restricted to the form

$$H_0: \quad p(\underline{x}_0) \geq p_0$$

versus

$$H_1: \quad p(\underline{x}_0) < p_0.$$

To test the null hypothesis, H_0, against the alternative hypothesis, H_1, compute the posterior probability that $\tilde{p}(\underline{x}_0) < p_0$, and form the ratio

$$Pr[\tilde{p}(\underline{x}_0) \geq p_0|y]/Pr[\tilde{p}(\underline{x}_0) < p_0|y].$$

This represents the posterior odds ratio of H_0 to H_1. Suppose a criterion value of Ω is used for decision-making. If the odds ratio is less than

MODEL

	A		B	
Sample Size	100	1000	100	1000
True $p(x_0)$	0.1104	0.1104	0.0165	0.0165
Maximum Likelihood Estimate (M. L. E.)	0.0578	0.0970	0.0127	0.0123
n'	10	10	10	10
r'	2	2	1	1
Bayesian Estimate	0.0894	0.1008	0.0239	0.0134
Effective Sample Size	35	264	68	814
98% M. L. E. Confidence Interval	(0.0113, 0.2489)	(0.0621, 0.1484)	(0.0010, 0.1366)	(0.0059, 0.0254)
98% Bayesian Credible Interval	(0.0192, 0.2107)	(0.0629, 0.1530)	(0.0019, 0.0862)	(0.0058, 0.0245)
Length of M. L. E. Confidence Interval	0.2376	0.0863	0.1356	0.0195
Length of Bayesian Credible Interval	0.1915	0.0901	0.0843	0.0187

Figure 1: Examples of Bayesian Credible Intervals and Corresponding Maximum Likelihood Estimate Confidence Intervals

Ω, H_0 is rejected; if not, then H_0 cannot be rejected based on the value Ω. The probabilities used to form the above odds ratio can be approximated using a beta distribution with parameters $r'' = N^* \hat{p}(\underline{x}_0) + r'$ and $n'' = N^* + n'$. Figure 2 contains a chart of hypothesis testing examples using models A and B with sample sizes of 100 and 1000 for each. The critical $\underline{x}_0$ vector used in these examples was:

$$\underline{x}_0 = (-0.7770, \ -0.3436, \ 0.7804, \ 0.5683, \ 0.0975, \ -0.5476).$$

C. CRITICAL COMBINATIONS OF THE VARIABLES

An inference of much importance in this study is the assessment of $p(\underline{x}_0)$, the true probability of injury for a given set of conditions $\underline{x}_0$, in relation to some small probability, p_0. Such inferences can be made by the prediction of critical envelopes and by testing hypotheses concerning $p(\underline{x}_0)$. To see how to best use Bayesian hypothesis tests for making inferences as stated above, see [2] and substitute Bayesian tests for the classical procedures given there.

A critical envelope can be defined as the set of all combinations of independent variables for which the predicted probability of injury equals some given value. However, in this case variability in the predicted probability causes variability in the prediction of the critical envelope. To predict safer critical envelopes, such variability must be accounted for. As shown in [2], a safer critical envelope can be formed from the set of $\underline{x}$ vectors such that the upper end point of a $100(1 - \alpha)\%$ right-sided confidence interval for $p(\underline{x})$ equals p_0. A Bayesian analogy of this is formed by using Bayesian credible intervals

	MODEL			
	A		B	
Sample Size	100	1000	100	1000
True $p(\underline{x}_0)$	0.2813	0.2813	0.0503	0.0503
Maximum Likelihood Estimate	0.3229	0.2493	0.0438	0.0320
Bayesian Estimate	0.2737	0.2460	0.0575	0.0338
n'	10	10	10	10
r'	2	2	1	1
Effective Sample Size	15	139	31	367
Null Hypothesis	$p(\underline{x}_0) \geq .3$	$p(\underline{x}_0) \geq .3$	$p(\underline{x}_0) \geq .1$	$p(\underline{x}_0) \geq .1$
Alternative Hypothesis	$p(\underline{x}_0) < .3$	$p(\underline{x}_0) < .3$	$p(\underline{x}_0) < .1$	$p(\underline{x}_0) < .1$
Posterior Probability of Null Hypothesis	.3783	.0779	.0623	.0000+
Bayesian Posterior Odds Ratio	.6085	.0845	.0664	.0000+
Classical Hypothesis Test p-value	.5753	.0968	.1563	.0000+
Criterion Value Ω	.05	.05	.05	.05
Significance Level α	.05	.05	.05	.05
Classical Hypothesis Test Conclusion	fail to reject H_0	fail to reject H_0	fail to reject H_0	reject H_0
Bayesian Test Conclusion	fail to reject H_0	fail to reject H_0	fail to reject H_0	reject H_0

Figure 2: Some Hypothesis Testing Examples

to incorporate sample and prior information into the prediction of a critical envelope.

An exact Bayesian critical envelope requires prior estimates of r' and n' for each $\underline{x}_0$. These estimates are likely to be difficult to acquire. However, it is reasonable to assume that the prior distribution for $\tilde{p}(\underline{x})$ is the same for all $\underline{x}$'s in the critical envelope. A practical choice for a prior distribution is a beta distribution with parameters r' and n' such that $(r'/n') = p_0$. The value of n' can be chosen to represent the strength of the experimenter's prior convictions. For example, if the experimenter feels that his or her prior information is worth ten sample data observations, then the choice would be n' = 10.

To compute a right-sided Bayesian $100(1 - \alpha)\%$ credible interval for $p(\underline{x})$, a number c must be found such that

$$Pr[0 \le \tilde{p}(\underline{x}) \le c|y] = 1 - \alpha.$$

Here $\tilde{p}(\underline{x})$ has a beta distribution with parameters r" and n" which, as defined previously, incorporate both the prior and sample information. To construct a Bayesian critical envelope, all $\underline{x}$'s which satisfy $c = p_0$ must be found. Methods of numerical analysis are required since c, as a function of $\underline{x}$, has no closed functional form.

Although there is no closed functional form for the equation describing the Bayesian critical envelope, it is possible, using a digital computer, to find an approximate closed form for the Bayesian critical envelope. To do this, first store (in core or on data records) a grid of points roughly spanning the $\underline{x}$-space. For each point on the grid, compute the corresponding upper end point of the right-sided $100(1 - \alpha)\%$ Bayesian credible interval. Then use multiple regression to find a best fitting prediction equation

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for the credible interval end point. Let $p_u(\underline{x})$ be the upper end point of the right-sided $100(1 - \alpha)\%$ credible interval for $p(\underline{x})$. Then the approximate equation

$$p_u(\underline{x}) \tilde{=} b_0(\alpha) + \sum_{i=1}^{m} b_i(\alpha) z_i(\underline{x})$$

results from the regression analysis, where $b_i(\alpha)$, $i = 0, 1,\ldots,m$ are the regression coefficients and where $z_i(\underline{x})$ is the $i^{\underline{th}}$ term of the best fitting multiple regression equation. Notice that because $p_u(\underline{x})$ depends upon α, the b_i's must also depend upon α. Once the regression equation has been found, set $p_u = p_0$. An approximate Bayesian critical envelope is then given by

$$c_\alpha(p_0) = \{x : p_0 = b_0(\alpha) + \sum_{i=1}^{m} b_i(\alpha) z_i(\underline{x})\}.$$

This approximate critical envelope yields a simple closed functional form and is thus useful for theoretical purposes.

A revealing way to study a critical envelope is to make 2-dimensional plots of one of the x_i variables versus one of the other x_i variables with p_0 and the other x_i variables fixed at given values. This was done in a previous Desmatics technical report [2] for the classical inference situation. The approximate or exact Bayesian critical envelope may be used to create these plots. For example, with p_0 and $x_3,\ldots,x_6$ satisfy the critical envelope equation. Then it can be graphically seen how x_1 and x_2 should vary to keep the probability of injury below p_0 for fixed $x_3,\ldots,x_6$. Obviously to plot x_1 versus x_2 one must assign values to x_2, say, and then solve for x_1. For the exact Bayesian critical envelope a numerical analysis computer routine would be needed to do this since no closed functional form

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exists. The approximate Bayesian critical envelope may also require the use of a numerical analysis routine depending upon the complexity of the resulting multiple regression equation.

V. SUMMARY AND DISCUSSION

The incorporation of prior information, through the use of Bayesian techniques, may improve upon the inference procedures described in a previous technical report [2]. It should be recognized though, that large differences between the Bayesian and maximum likelihood procedures may be due to bad sample information or gross misspecification of prior information. Thus, if large differences occur, both information sources should be rechecked thoroughly. Further improvement in inference procedures may possibly be achieved by utilizing preinjury data. This is briefly discussed in [3].

In constructing inference procedures for the assessment of injury probability, all available information should be used. The utilization of more sample information can be based on observed preinjury data. As shown in [3], preinjury data should increase the accuracy of the logistic regression coefficient estimates. This aspect of the injury probability prediction research is extremely important, since it is imperative that the most useful information be gained without subjecting human subjects to injury. Research on methods of employing preinjury data is now under way.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Research on the development of an impact acceleration injury prediction model based primarily on information in the data has been discussed in previous technical reports. This technical report describes Bayesian inference procedures which permit the experimenter to combine his or her prior beliefs and expert knowledge with the data observed in the experiment.

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